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HOW DOES PRE-SERVICE MATHEMATICS TEACHER PROVE THE LIMIT OF A FUNCTION BY FORMAL DEFINITION?

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Abstract

The purpose of this study was to investigate the flow of thought of the pre-service mathematics teachers through the answers of a function limit evaluation by formal definition. This study used a qualitative approach with descriptive method. The research subjects were the students of mathematics education department of Universitas Serang Raya, Indonesia. After analyzing the students' written answers, we interviewed the subjects to get further explanation on their strategies and common mistakes. This study found that based on the students' results in the function limit evaluation by formal definition, there were common strategies, i.e. (1) preparing the proof and (2) proving. The stage of preparing the proof consisted of (1) determining delta value by the final statement of formal definition, (2) substituting the given *f(x)* and *L* process, (3) simplifying value in the absolute sign, (4) solving the inequality, and (5) finding the delta value. The stage of proving consisted of (1) stating positive epsilon, (2) defining delta, (3) stating positive delta, (4) substituting the constants and delta values in the initial statement of formal definition, and (5) solving the inequality to create the final inequality statement of the formal definition.

Keywords: Formal definition of limit, Limit of function, Pre-service mathematics teacher, Proving limit strategy.

Abstrak

Tujuan penelitian ini adalah untuk menginvestigasi alur pikir mahasiswa calon guru matematika melalui jawaban soal mengevaluasi limit suatu fungsi dengan menggunakan definisi formal. Penelitian ini menggunakan pendekatan kualitatif dengan metode deskriptif. Subjek penelitian ini adalah mahasiswa calon guru matematika di program studi pendidikan matematika Universitas Serang Raya. Setelah menganalisis jawaban tertulis mahasiswa, wawancara dilakukan untuk mendapatkan penjelasan lebih lanjut mengenai strategi mereka dan kesalahan umum yang dilakukannya. Penelitian ini menemukan bahwa berdasarkan hasil pekerjaan mahasiswa dalam menjawab soal evaluasi limit fungsi dengan definisi formal, terdapat strategi umum yang dapat teridentifikasi, yaitu (1) tahap persiapan pembuktian dan (2) tahap pembuktian. Tahap persiapan pembuktian terdiri dari (1) menentukan nilai delta melalui pernyataan akhir definisi formal, (2) proses substitusi, (3) menyederhanakan nilai dalam tanda mutlak, (4) proses penyelesaian pertidaksamaan, dan (5) proses menemukan nilai delta. Tahap pembuktian meliputi (1) pernyataan epsilon positif, (2) mendefinisikan delta, (3) pernyataan delta positif, (4) proses substitusi nilai konstanta dan delta pada pernyataan awal definisi formal, dan (5) proses penyelesaian pertidaksamaan hingga membentuk pertidaksamaan pada pernyataan akhir definisi formal.

Kata kunci: Definisi formal limit, Limit Fungsi, Mahasiswa calon guru matematika, Strategi pembuktian limit.

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Calculus is the early course towards advanced mathematics which is taken by undergraduate students in most exact science and engineering department. It also provides the main path of deductive reasoning. Therefore, it is very important for students to have the accurate concept on the main idea of mathematics in this course, including the high-level generalization and abstraction (Mokhtar *et al.*, 2010; Oktaviyanthi & Supriani, 2015).

One of the basic concepts of calculus is limit. Teaching and learning the concept of limit have been important and interesting research subjects in mathematics education. The results of the study on students' understanding on the concept of limit show that students have delicate conceptual representation on the limit (Davis & Vinner, 1986; Karatas *et al.*, 2011; Beynon & Zollman, 2015). It is problematic for most students to apply the concept of limit intuitively into a formal concept. Many students do not understand the formal definition of a limit as a statement equal to what they have learned intuitively (Row, 2007; Kim *et al.*, 2015).

The result of a preliminary study found that the difficulty of most students in evaluating limit by formal form is related to the understanding of the concept of the formal definition of limit. The students do not clearly understand the definition of this concept and they often make a partial understanding of the definition based on several examples. This phenomenon prevents them from solving new problems different from the ones they have ever solved. It was also found that definition and theorem had been fragmented in the students' minds as something absolute and do not need to be proven again. This fact showed that they were not used to thinking that they needed to prove it as the basis of a generalization of a proposition and checking evidence, so it was difficult for them to see the central idea of a formal definition. This result was closely related to the students' habit to not pay attention to the theoretical aspect. They only memorize algorithmic procedures without understanding the application.

In advanced mathematics in the university, students have to master proving techniques and logical argument besides having the algebraic procedural ability. Proving is an advanced mathematics skill which is considered the most difficult skill to achieve by most students (Moore, 1994; Tall, 1998; Astawa *et al*, 2018). A study shows that the students' proving ability is still low (Schwarz & Kaiser, 2009; Hendroanto, et al. 2018; Shahrill, et al. 2018). There are many factors including the difficulty of proving skill, including learning experience, as stated by Moore (1994) that experience in constructing proof is limited to basic skill. Another reason for students' failure in proving according to Weber (2010) is the lack of strategic knowledge. Meanwhile, Selden & Selden (2003) state that students cannot determine whether the particular proof is valid or not.

The present study investigated the results of the students' work in solving limit problems by using formal form, especially how to construct formal proof and explore common mistakes. The purpose of this study was to determine the strategies of 20 mathematics education department students when evaluating the limit by formal definition. This study looked for an explanation of the students' strategies and their common mistakes when solving the problems. Therefore, this study presented students' written results described as their strategies. Moreover, this study also discussed students' responses to questions related to their solving strategies and mistakes.

METHOD

To achieve the research purpose, we used a qualitative approach with a descriptive explorative method to collect the data. This method refers to Creswell (2014) who describes the characteristics of a qualitative study to have a natural situation and to describe the actual result produced by the research subjects. Therefore, we did not give any treatment, action, or manipulation on the subjects. Thus, the data was original and we could obtain a lot of information on the research subjects' view on the discussion topic. We analyzed the students' work results by the concept of the formal definition of limit.

Research Subject

We used a purposive sampling method to select the 20 undergraduate students taking the calculus course in Universitas Serang Raya as the research participants. These subjects had varying initial mathematics skills. Furthermore, we interviewed nine students based on their test results. We selected the interviewees based on the results of a written test showing similar solving strategies and by detecting mistakes and misconceptions. Participants who showed different approaches were selected to enrich the results.

Limit Proving Question

This study used two limit evaluation questions by formal definitions given at two different times. The procedure was performed to see the consistency of the research subjects' answers.

Research Procedure

This study combined two data collection techniques, i.e. mathematics test containing limit evaluation questions with formal definition and interview. Therefore, the research data was categorized into two types, i.e. written and oral. We administered the mathematics test to the 20 mathematics education students in Universitas Serang Raya twice with two-week pause between them. We asked them to answer the questions in 30 minutes and then analyzed them. From the results of the analysis test answers, there are four answers which were most representative and closest to the procedure of evaluating the limit by formal definition.

After analyzing the answers of the written test, we interviewed the research subjects to get in-depth information and subjects' views on the answers of mathematics test by focusing on their strategies and concept when answering the questions. The interview method in this study was a task-based interview which is an interview conducted on a subject or a group of subjects by performing a task or finishing a mathematics test (Maher & Sigley, 2014). The mathematics test used in the interview was the same as the previously written test so that the research subjects could provide an in-depth explanation of their answers. To aid the interview, we prepared question guide before the interview.

Data Analysis

We compiled the results of the research subjects' written test and used all the answer sheets. All research subjects answered and finished the questions. The respondents' answers selected for interview were labeled R1 to R9. It's to make it easier for us to identify and analyze their results. The next step was the in-depth analysis of each stage of problem-solving.

In general, the strategies to solve limit evaluation by formal definition were determined by three consecutive stages. The first stage was evaluating every respondent's answers step by step. In the other words, we analyzed every step in detail to make sure that the respondents' answers in two different times had similar processes. We noted how respondents answered the questions, starting from the rule or theorem they used, argument and work step they chose, and technique to write down answer they applied. The second stage is categorization, meaning compiling the answering processes of respondents who had similar strategies and labeling them. The third step is attaching the respondents' explanation on the actual rationalization of each step of their answers and the examples based on the respondents' perspective which is obtained using the interview.

By still considering respondents' common mistakes, we analyzed the mistakes by evaluating every answer step by step based on the concept of the formal definition of limit. Afterward, we listed the respondents' mistakes and classified them generally. Then, we tried to get additional information on the respondents' mistakes through actual argumentation and examples from the respondents.

The determination of strategy to solve the problems and the respondents' mistakes were categorized by the answer to the written test. Therefore, the categories were obtained inductively. Possible categories were discussed to get general categories. Three mathematics education experts were consulted to get a result which was consistent with the agreement and did not violate any rule.

In this study, the principle of truth, valid and objective, can be described by credibility, transferability, dependability, and confirmability (Guba & Lincoln, 1982). Every detail of test answers described by the respondents, including interview and other data collection procedures, was described to measure from various perspectives of the respondents to get a valid result. Interview data was recorded so that the potential of information loss could be avoided. The research subject determination by purposive sampling and in-depth description were expected to make the research result can be applied in general and can be transferred or used in different contexts or more specific contexts.

This study used various data sources and collection techniques, i.e. written mathematics test, interview, interview-based test, and literature study. The triangulation was performed to meet data validity criteria (Creswell, 1998). Additionally, our discussion with mathematics education experts was performed after triangulation to check the researcher's interpretation and support data validity.

RESULTS AND DISCUSSION

The analysis result provided valuable information on respondents' strategies and common mistakes when solving limit evaluation questions by the formal definition of limit. Therefore, the researcher separated the discussion into two aspects based on the focus of this study. Everything will be explained by presenting the main data from the respondents' written answers and interview. Three concepts of the formal definition of limit are included for basic discussion in detailed descriptions of some of the respondents' common mistakes.

Students' Strategy

The respondents' strategies to solve limit evaluation problems by the formal definition of limit, in general, differed depending on their skill. The respondent with more experience of proving problems, especially by formal definition, showed effective solution and gave logic argumentation confidently. Meanwhile, the respondents with no experience or could not use formal rule tended to guess the answers, used theories or rules incorrectly, and even did not answer the questions. It was in line with Row (2007) who states that the activity of understanding the concept of limit requires experience with mathematically rigorous processes.

The basic analysis of the respondents' written test resulted in the summarized general strategies shown by various ways of respondents problem-solving, namely (1) preparation of proof and (2) proving. The preparation of proof consisted of (1) determining delta value by the final statement of formal definition, (2) substitution process, (3) simplifying value in absolute sign, (4) solving inequality, and (5) finding delta value. The proving consisted of (1) positive epsilon statement, (2) defining delta, (3) positive delta statement, (4) substitution of constants and delta values in the initial statement of formal definition, (5) solving the inequality to create the final statement inequality of the formal definition. We found the strategies from the respondents who presented the solutions correctly and consistently with the rules.

Phrase in Formal Definition of Limit

In the formal definition of limit, there are at least five phrases to understand, i.e.:

1. The phrase "for every positive epsilon ($\forall \varepsilon > 0$)" means that we do not have control over epsilon and thus the proving must apply for every epsilon.
2. The phrase "there is a positive delta ($\exists \delta > 0$)" means that the proving must produce delta value. So, the existence of the number is confirmed. Specifically, delta value will depend on the epsilon value.
3. The phrase "in such a way that for every x ($\ni \forall x$)" means that we cannot limit x value more than the available limit.
4. The phrase "if $0 < |x - c| < \delta$" is an initial statement (antecedent) in the implication part in the formal definition of limit. The expression $|x - c| < \delta$ means that x value will approach c, but not exceeds (nor is equal to) delta value. The expressiom $0 < |x - c|$ means that x does not equal to c.

5. The phrase "so $|f(x) - L| < \varepsilon$" is the conclusion statement (consequent) in the implication part of the formal definition of limit. When the statement is met, the proving process is finished.

The key in proving limit by using formal definition is the identification of delta value. To find the delta value, the limit evaluation uses formal definition starting from the conclusion statement $|f(x) - L| < \varepsilon$ and working backward until it reaches expression equivalent with $|x - c| < \delta$. Additional explanation using narrative to connect mathematics statements tremendously helps understanding. As in the study by Tall (1998), using sentences or narration in proving is the student's personal imagery which can help them forming or producing a formal argument. This type of proving is called constructive proof, which is a part of deductive proving. The respondents' strategies to identify delta value is presented in the Figure 1, Figure 2, Figure 3, and Figure 4.

- Show that every $\varepsilon > 0$ there is $\delta > 0$.
- Take $\varepsilon > 0$ so $\left|\left(\frac{2(x^2-4)}{x-2}\right) - 8\right| < \varepsilon$ (1) can be written when $0 < |x - 2| < \delta$ (2)
- δ value depends on ε, so determine the relation between absolute value $\left|\left(\frac{2(x^2-4)}{x-2}\right) - 8\right|$ and $|x - 2|$, which is:

Figure 1. The strategy of Respondent 1 in identifying the δ value

For every $\varepsilon > 0$, there is $\delta > 0$, so:
If $0 < |x - 2| < \delta$, then $\left|\left(\frac{2(x^2-4)}{x-2}\right) - 8\right| < \varepsilon$
It means when x approaches 2, there's always small number δ ($\pm\delta$) in the range of 2 so there's also always small number ε ($\pm\varepsilon$) in the range of 8 which causes the value of f(2) to approach 8.
Therefore, the relation between ε and δ can be written as:

$$L \pm \varepsilon = f(c \pm \delta) \equiv 8 \pm \varepsilon = \frac{2((2 \pm \delta)^2 - 4)}{(2 \pm \delta) - 2}$$

Figure 2. The strategy of Respondent 2 in identifying the δ value

To use formal definition in proving, the process starts from $0 < |x - 2| < \delta$ to $\left|\left(\frac{2(x^2-4)}{x-2}\right) - 8\right| < \varepsilon$, meaning the formula of δ (in relation with ε) must be found correctly.
Because the value of δ depends on ε, the proving process starts from $\left|\left(\frac{2(x^2-4)}{x-2}\right) - 8\right| < \varepsilon$. Simplifies the form:

Figure 3. The strategy of Respondent 3 in identifying the δ value

- Prepare proving: if $0 < |x - 2\ | < \delta$, the $\left|\left(\frac{2(x^2-4)}{x-2}\right) - 8\right| < \varepsilon$
- Find required factors
  $|x - c| \Rightarrow |x - 2\ |$ and $|f(x)L| \Rightarrow \left|\left(\frac{2(x^2-4)}{x-2}\right) - 8\right| \Rightarrow 2|x - 2|$
- Choose delta: because $\left|\left(\frac{2(x^2-4)}{x-2}\right) - 8\right| \Rightarrow 2|x - 2| < \varepsilon \Rightarrow |x - 2| < \frac{\varepsilon}{2}$ and $|x - 2\ | < \delta$, so choose $\delta = \frac{\varepsilon}{2}$

Figure 4. The strategy of Respondent 4 in identifying the δ value

The four respondent strategies for evaluating limit value by formal definition generally had the same flow, i.e. starting by identifying δ value. The first, third and fourth respondents identified δ value from the consequent statement in the implication part in the formal definition of limit and this strategy is common. Meanwhile, the second respondent used a different strategy by interpreting delta and epsilon as small values in the range of x and $f(x)$. It is quite logical, but not common in formal proving. The second respondent argued that they used delta and epsilon interpretation to start proving based on a geometric picture of the positions of delta and epsilon they remembered. Respondent 2 said, "*it was easier for me to solve this question by imagining the positions of delta and epsilon in Cartesian coordinates. Delta in the range of x, and if x has an image f(x), there is another value near f(x) which is a map of delta called epsilon*." Each respondent gave a different narration in proving based on their understanding.

Algebraic Technique

Proving by formal definition starts from antecedent statement to consequent. Implicitly, this type of problem solving goes through two stages. The first stage is guessing delta value and the second stage is showing that the usage of the delta value is correct so the proving is correct. Absolute algebra is used in both processes without ignoring the rules in effect in proving by formal definition. The respondents used algebraic techniques in proving as shown in Figure 5, Figure 6, Figure 7, and Figure 8.

- The value of δ depends on ε, so determine the relation between absolute value $\left|\left(\frac{2(x^2-4)}{x-2}\right) - 8\right|$ and $|x - 2|$, which is:

$$\left|\left(\frac{2(x^2-4)}{x-2}\right) - 8\right| = \left|\left(\frac{2(x-2)(x+2)}{x-2}\right) - 8\right| = |2(x+2) - 8| = 2|x - 2| \qquad \dots (3)$$

- From (1), (2) and (3), $\left|\left(\frac{2(x^2-4)}{x-2}\right) - 8\right| = 2|x - 2| \equiv \varepsilon = 2\delta \equiv \delta = \frac{\varepsilon}{2}$ is found
- So if $\varepsilon > 0$ is taken, $\delta = \frac{\varepsilon}{2}$ can be selected. To check whether the selected δ yais correct, use $0 < |x - 2| < \delta = \frac{\varepsilon}{2}$
  to produce $\left|\left(\frac{2(x^2-4)}{x-2}\right) - 8\right| = 2|x - 2| < 2.\frac{\varepsilon}{2} = \varepsilon$
- So $\lim_{x\to 2}\left(\frac{2(x^2-4)}{x-2}\right) = 8$ is correct.

Figure 5. The algebraic technique of Respondent 1

Therefore the relation between ε and δ can be written as:

$$L \pm \varepsilon = f(c \pm \delta) \equiv 8 \pm \varepsilon = \frac{2((2 \pm \delta)^2 - 4)}{(2 \pm \delta) - 2}$$

$$\equiv 8 \pm \varepsilon = \frac{2\big((2 \pm \delta) - 2\big)\big((2 \pm \delta) + 2\big)}{(2 \pm \delta) - 2} \equiv 8 \pm \varepsilon = 2\big((2 \pm \delta) + 2\big)$$

$$\equiv 8 \pm \varepsilon = 2\big((2 \pm \delta) + 2\big) \equiv 8 \pm \varepsilon = 8 \pm 2\delta \equiv \pm\varepsilon = \pm 2\delta \equiv \varepsilon = 2\delta \equiv \delta = \frac{\varepsilon}{2}$$

Therefore producing $\delta = \frac{\varepsilon}{2}$. To check that the selected delta value is correct, use:

$$L \pm \varepsilon = f(c \pm \delta) \equiv 8 \pm \varepsilon = \frac{2((2 \pm \delta)^2 - 4)}{(2 \pm \delta) - 2}$$

$$\equiv 8 \pm \varepsilon = \frac{2\big((2 \pm \delta) - 2\big)\big((2 \pm \delta) + 2\big)}{(2 \pm \delta) - 2} \equiv 8 \pm \varepsilon = 2\big((2 \pm \delta) + 2\big)$$

$$\equiv 8 \pm \varepsilon = 2\big((2 \pm \delta) + 2\big) \equiv 8 \pm \varepsilon = 8 \pm 2\delta$$

$$\equiv 8 \pm \varepsilon = 8 \pm 2\left(\frac{\varepsilon}{2}\right) \equiv 8 \pm \varepsilon = 8 \pm \varepsilon$$

Because $L \pm \varepsilon = f(c \pm \delta)$, so $\lim_{x \to 2}\left(\frac{2(x^2-4)}{x-2}\right) = 8$ is correct.

Figure 6. The algebraic technique of Respondent 2

Because δ value depends on ε, proving process starts from $\left|\left(\frac{2(x^2-4)}{x-2}\right) - 8\right| < \varepsilon$ (1), simplifies the form:

$$\left|\left(\frac{2(x^2-4)}{x-2}\right) - 8\right| < \varepsilon = \left|\left(\frac{2(x-2)(x+2)}{x-2}\right) - 8\right| < \varepsilon = |2x + 4 - 8| < \varepsilon$$

$$= |2x - 4| < \varepsilon = 2|x - 2| < \varepsilon \equiv |x - 2| < \frac{\varepsilon}{2}$$

After simplifying inequality (1) to produce inequality $|x - c| < \delta$, in this case assuming the value of $\delta = \frac{\varepsilon}{2}$, and proving can be written as: given ε, assuming $\delta = \frac{\varepsilon}{2}$, then:

$$|x - c| < \delta \Rightarrow |x - 2| < \frac{\varepsilon}{2} \Rightarrow 2|x - 2| < \varepsilon \Rightarrow |2x - 4| < \varepsilon \Rightarrow |2x + 4 - 8| < \varepsilon$$

$$\Rightarrow \left|\left(\frac{2(x-2)(x+2)}{x-2}\right) - 8\right| < \varepsilon \Rightarrow \left|\left(\frac{2(x^2-4)}{x-2}\right) - 8\right| < \varepsilon \Rightarrow |f(x) - \mathrm{L}| < \varepsilon$$

Because the assumed delta value is correct and produces correct consequent statement, $\lim_{x \to 2}\left(\frac{2(x^2-4)}{x-2}\right) = 8$ is correct and proving is finished.

Figure 7. The algebraic technique of Respondent 3

- Find required factors
  $|x-c| \Rightarrow |x-2\ |$ and $|f(x)L| \Rightarrow \left|\left(\frac{2(x^2-4)}{x-2}\right)-8\ \right| \Rightarrow 2|x-2|$
- Select delta: because $\left|\left(\frac{2(x^2-4)}{x-2}\right)-8\ \right| \Rightarrow 2|x-2| < \varepsilon \Rightarrow |x-2| < \frac{\varepsilon}{2}$ and $|x-2\ | < \delta$, so select $\delta = \frac{\varepsilon}{2}$
- Show that the selected delta is correct
  Given $\varepsilon > 0$, assuming $\delta = \frac{\varepsilon}{2}$, then

$$|x-2\ | < \delta \Rightarrow |x-2\ | < \frac{\varepsilon}{2} \Rightarrow |x-2\ | < \frac{\varepsilon}{2} \Rightarrow 2|x-2| < \varepsilon \Rightarrow |2x-4| < \varepsilon$$

$$\Rightarrow |2x+4-8\ | < \varepsilon \Rightarrow \left|\left(\frac{(2x+4)(x-2)}{x-2}\right)-8\ \right| < \varepsilon$$

$$\Rightarrow \left|\left(\frac{2x^2-4x+4x-8}{x-2}\right)-8\ \right| < \varepsilon \Rightarrow \left|\left(\frac{2x^2-8}{x-2}\right)-8\ \right| < \varepsilon$$

$$\Rightarrow \left|\left(\frac{2(x^2-4)}{x-2}\right)-8\ \right| < \varepsilon \Rightarrow |f(x)-\mathrm{L}\ | < \varepsilon$$

Because if $0 < |x-2\ | < \frac{\varepsilon}{2}$ it means that $|f(x)-8\ | < \varepsilon$, proving is finished.

Figure 8. The algebraic technique of Respondent 4

Respondent 1 believed that delta depends on epsilon, so the Respondent 1 connected the statement containing epsilon element with the statement containing delta. The algebraic technique started from the expression $|f(x)-L| = \left|\left(\frac{2(x^2-4)}{x-2}\right)-8\ \right|$ which was formed in such a way which produced expression equivalent with $|x-c| = |x-2|$ until delta value was found in epsilon variable. Then, the delta value was substituted in the initial expression to check the correctness.

Similar with Respondent 1, Respondent 2 also believed that delta value is related to epsilon value. Respondent 2 used the understanding that delta and epsilon are small values in the range of x and $f(x)$, so if x corresponds with $f(x)$, then delta corresponds with epsilon. If x approaches c, $f(x)$ approaches L, it means if there is a delta value in the range of x which approaches c, there is epsilon value in the range of $f(x)$ approaching L. The algebraic technique was in the expression $L = f(x) \Rightarrow L \pm \varepsilon = f(c \pm \delta)$, producing delta value. Then, Respondent 2 checked the correctness of the value by substituting it back to the initial expression, producing $8 \pm \varepsilon = 8 \pm \varepsilon$. The result was claimed to be correct by Respondent 2 and they concluded that $\lim_{x\to 2}\left(\frac{2(x^2-4)}{x-2}\right) = 8$ was correct.

The proving of Respondent 2 is not common. Although it was started by identifying delta value and obtaining the correct result, Respondent 2 did not use common proving in implication type of relation. Tall (1998) calls it extracting meaning unsuccessfully. Respondent 2 used understanding that delta is a small value in the range of x when x approaches c, expressed as $c+\delta$ as $c-\delta$, which are actually the definition of the expression $|x-\mathrm{c}\ | < \delta$. Similarly, epsilon is a small value in the range of $f(x)$ when $f(x)$ approaches L, expressed as $L+\varepsilon$ and $L-\varepsilon$, which are actually the definition of the expression $|f(x)-\mathrm{L}\ | < \varepsilon$.

Researcher (R) : *You solved this by writing this equation $L \pm \varepsilon = f(c \pm \delta)$ when both looking for delta and checking whether the delta you chose is correct. Why did you answer this way?*

Respondent 2 (R2) : *Because that's how I understand it. The concept of limit with the definition that I understand uses this graph.* (R2 was drawing a graph and showing it to the researcher)

R : *Do you think your process is proving by formal definition?*

R2 : *All I know is proving by formal definition has delta and epsilon, so I think my method is a formal proving.*

The process done by Respondent 2 did not imply formal proving although they used elements of delta and epsilon. The understanding of Respondent 2 is called non-standard analysis by Tall (1992). In the formal proving, a direct evidence was used, i.e. proving that if p then q and it is done by assuming p then showing that it applies to q. Respondent 2's method is interesting to study, but a detailed discussion of it will be presented in another paper because it requires further research.

Respondent 3 revealed the same thing as another respondent that delta value depends on epsilon value. Delta value identification by Respondent 3 started from the consequent statement in the formal definition which was $\left|\left(\frac{2(x^2-4)}{x-2}\right) - 8\right| < \varepsilon$. The algebraic technique of Respondent 3 was seen in the simplification of inequality of $\left|\left(\frac{2(x^2-4)}{x-2}\right) - 8\right| < \varepsilon$ until it produced $|x-2| < \frac{\varepsilon}{2}$. The produced inequality equivalent with the antecedent statement in the formal definition which was $|x-2| < \delta$. So, Respondent 3 wrote $\delta = \frac{\varepsilon}{2}$. Then, Respondent 3 used the delta by substituting it into the antecedent statement $|x-c| < \delta \Rightarrow |x-2| < \frac{\varepsilon}{2}$ and continued until it produced $\left|\left(\frac{2(x^2-4)}{x-2}\right) - 8\right| < \varepsilon \Rightarrow |f(x) - \mathrm{L}| < \varepsilon$. In the proving stage, Respondent 3 seemed to move backward from $|x-2|$ to $\left|\left(\frac{2(x^2-4)}{x-2}\right) - 8\right|$. The algebraic technique was acceptable and met implication proving process, which is first assuming p (in this case, $|x-2| < \delta$) and showing that it also applies to q (in this case, $\left|\left(\frac{2(x^2-4)}{x-2}\right) - 8\right| < \varepsilon$).

The Respondent 4's work was substantially similar to the Respondent 3's work. It started by identifying delta value through the consequent statement and working backward by substituting delta value into the antecedent statement. The differences between their methods were the steps and narrative sentences. Respondent 4 provided the shorter steps and narration than Respondent 3 but was not as detailed as Respondent 3. They only mentioned main elements which might be required in proving without giving any additional explanation. Respondent 4 started solving the question by listing required factors which are $|x-2|$ and $\left|\left(\frac{2(x^2-4)}{x-2}\right) - 8\right|$ then simplifying them. The second step was selecting delta by associating consequent and antecedent statements to produce $\delta = \frac{\varepsilon}{2}$. Then, they used delta value by substituting it with the antecedent statement and working backward until it produced $|f(x) - \mathrm{L}| < \varepsilon$.

Common Mistake of Student

Formal Definition of Limit

Most respondents did not understand the mathematics statements which became important elements in the formal definition of limit, especially $0 < |x - c| < \delta$ and $|f(x) - L| < \varepsilon$. Nearly all questions related to limit evaluation by formal definition require complete understanding of implication logical relation, and inability to understand it was one of the reasons of the respondents failed in proving. In the implication, there are two propositions which are often symbolized as p and q. Proposition p is called antecedent and proposition q is called consequent. In the formal definition of limit, the antecedent is $0 < |x - c| < \delta$ and the consequence is $|f(x) - L| < \varepsilon$. These parts were the main points in solving the limit evaluation questions by formal definition.

Solving limit evaluation by formal definition uses direct evidence. It means proving that if p then q. It is done by assuming p then showing that it applies to q. The students' common mistake in solving limit questions by formal definition was the early stage of proving when determining the relations between delta, epsilon, $|x - c| < \delta$ and $|f(x) - L| < \varepsilon$.

R : *Why did you only do this proving question until the mathematics expression* $2|x - 2| < \varepsilon$*?*
R5 : *I didn't know what to look for in the calculation* (thinking). *Oh, wait, I can make it like this* $|x - 2\ | < \frac{\varepsilon}{2}$ (R5 was writing the inequality and showing it to the researcher).
R : *After that form, what can you know further?*
R5 : (thinking for a long time and then shaking head) *I don't know. I'm confused.*

If Respondent 5 had understood the relations among delta, epsilon, and other statements in the formal definition, Respondent 5 would have been able to answer that from the form $|x - 2\ | < \frac{\varepsilon}{2}$, Respondent 5 would get delta value because the form was equal to $|x - 2| < \delta$ which contained delta. A lack of understanding on the relations caused another possible mistake which was "leap" in strategy, meaning loss of logical procedure by missing one or more steps, thus causing misconception. Students' leap because of the lack of understanding of concept was detected in the interview session. One of the leaping solution was performed by Respondent 6 as shown in Figure 9.

Given: $\varepsilon > 0, \delta > 0$
Known: $|x - 2\ | < \delta \Rightarrow -\delta < x - 2 < \delta \Rightarrow 0 < x - 2 < \delta \Rightarrow x - 2 < \delta$

$$\left|\left(\frac{2(x^2 - 4)}{x - 2}\right) - 8\ \right| < \varepsilon \Longrightarrow |2(x + 2) - 8\ | < \varepsilon \Longrightarrow |2x - 4| < \varepsilon$$

$$\Longrightarrow |2(x - 2)| < \varepsilon \Longrightarrow |2\delta| < \varepsilon \Longrightarrow \delta > \frac{\varepsilon}{2}$$

Figure 9. Respondent 6's work

R : *Can you explain the process of working on* $|2(x - 2)| < \varepsilon \Longrightarrow |2\delta| < \varepsilon \Longrightarrow \delta > \frac{\varepsilon}{2}$*?* (pointing to respondent 6's answer sheet). *Why did you substitute* $x - 2$ *into* δ*?*
R6 : *I got this* (pointing to $\mathrm{x} - 2$) *from this* (pointing to $|\mathrm{x} - 2\ | < \delta \Rightarrow -\delta < x - 2 < \delta \Rightarrow 0 < x - 2 < \delta \Rightarrow x - 2 < \delta$).

R : *Are you sure $|x-2|<\delta$ equals to $x-2<\delta$?*
R6 : (quiet for a moment) *Yes* (sounding unsure).
R : *Do you understand the process of proving the limit by formal definition?*
R6 : *Yes. Look for delta value first, then insert it into epsilon equation. But, I shortened it, from here* (pointing to $|x-2|<\delta$) *then I found delta value, and I changed $|x-2|$ here* (pointing to $|2(x-2)|<\varepsilon$) *with the delta I found, and I found the answer.*

Based on the Respondent 6's explanation, there was a conceptual mistake. The main concept of proving by looking for delta value first is correct, but Respondent 6 made a mistake in the concept of operating $|X-2|<\delta$ to produce delta. The student's leap could not be generalized as poor understanding. Some explained that the leap was for practicality or a form of creativity. This was implied in the interview session with Respondent 7.

R : *Why did you write $\left|\left(\frac{2(x^2-4)}{x-2}\right)-8\right|<\varepsilon \Rightarrow |x-2\ |<\frac{\varepsilon}{2}$?*
R7 : *Because it's obvious that it's the answer.* (then pointing with finger) *This part $\frac{2(x^2-4)}{x-2}$ when simplified will become $2x+4$, then reduced by 8. It's clear, it's for practicality, so I don't have to write too much.*

The students should understand that the main key to understanding the relations between phrases in the formal definition of a limit is delta value which depends on epsilon, the operating statement containing epsilon ($|f(x)-L|<\varepsilon$) in a way that forms statement equivalent with the statement containing delta ($|x-c|<\delta$). With the relation, delta value will be found as the basic assumption to continue to the next proving stage.

Algebraic Manipulation

Evaluating limit by formal definition signifies that we must find $\delta>0$ for every $\varepsilon>0$, where ε can be made as small as possible. We must prove that for every ε around L, there is (at least one) δ around c but does not include c, so inequality $|f(x)-L|<\varepsilon$ applies. Algebraic manipulation is required in solving the problem. We must start with $|x-c|<\delta$ and manipulating it until we reach $|f(x)-L|<\varepsilon$. The manipulation is to get the relation between delta and epsilon which is finding delta value in epsilon. Petersen (2010) states, "*95% of the time, you should start with your ε term and algebraically manipulate it until you get something of the form $|g(x)|.|x-c|<\varepsilon$*".

Several respondents showed good skill in solving limit evaluation by this formal definition. But some made mistakes in making an algebraic manipulation, even showing an inability to process the algebraic form. It inhibited the proving process. For example, the work process of Respondent 6 in finding delta value. There was a mistake in the manipulation process. Although the end result was correct, the overall proving process was failed. Respondent 6's mistake was in finding delta value, i.e. showing the process $|x-2\ |<\delta$ into $x-2<\delta$ which was then substituted into $|2(x-2)|<\varepsilon$ thus producing $|2\delta|<\varepsilon \Longrightarrow \delta>\frac{\varepsilon}{2}$.

Use of Additional Learning Source

The result of this study highlighted several strategies in evaluating limit by formal definition by education students to show their understanding of the concept of limit. Based on the result of the written test, at least 7 students followed the concept and finished it, 5 students partially finished the process and had trouble with algebraic manipulation, and the remaining 8 students were a combination of finishing questions not in accordance with the concept and not understanding the concept at all. Based on the interview with respondents who were able to solve the questions well, they used additional learning media beside textbooks and lectures to get a better understanding.

R : *Besides textbooks and lectures, what do you do to understand the concept of proving a limit by formal definition?*
R3 : *I browsed videos on Youtube to see examples of solving these types of questions.*
R : *Do the problem explanation on Youtube help you? Why?*
R3 : *It really helps me. In the Youtube videos, I can clearly see the stages. I also can download them so I can watch it repeatedly so that I understand.*
R : *Can you show me the video?*
R3 : *Oh, this one* (respondent showing the video).

R : *How do you understand the concept of evaluating the limit by formal definition, besides using textbooks and lectures?*
R1 : *I download tutorial videos from Youtube.*
R : *How did you find these videos?*
R1 : *I searched the keyword 'how to solve limit using formal definition', then there were many videos showing how to do it. I chose the easiest to understand, the ones with clear steps.*
R : *Do the videos help your understanding?*
R1 : *They really do, bu. I understand it a little. If I forget how to do it, I just have to replay them.*

The result of the interviews with Respondent 1 and Respondent 3 on additional learning sources indicated that using tutorial learning videos showing the stages to solve the questions or stages of solution helped their understanding in evaluating limit by formal definition. The respondents understood by replaying the lesson videos they downloaded. Using video in face-to-face class is becoming increasingly acceptable and keeps increasing in the past two or three decades in higher education, especially in the blended learning environment (Bennett & Maniar, 2007; Cruse, 2007; Ibrahim, 2012; Kinnari-Korpela, 2015). According to some studies, many students are helped in learning by this media. Kinnari-Korpela (2015) states that students with low mathematics skill show increase in understanding when learning in class is integrated with video usage. Moreover, Oktaviyanthi & Herman (2016) state in their study that there are at least five positive things in using video in learning, including helping students in making the concepts of Calculus are visible. In other words, using videos on how to solve limit evaluation by formal definition could help the respondents see how the components in the formal definition of limit work. Further studies are required on the utilization of lesson video in solving proving questions in general.

Making Proving Flow

The researcher found that one of the difficulties faced by the students in proving by formal definition was how to start or what step to take after the first step and so on. This is common in proving because students are generally used to routine questions which often rely on algebraic operations. Furthermore, sometimes textbooks do not explain the steps of proving in detail, so the students' mindset in processing proving was not facilitated. Perhaps the students understand the concept of the formal definition of limit in general as explained by their lecturers, but limited teaching method to develop theoretical mindset toward practical mindset caused difficulty in solving proving questions.

Oktaviyanthi & Herman (2016) say that the incorrect method in presenting a concept can be one of the causes why the students hard to recognize the concept. Similarly, Respondent 1 and Respondent 3 claimed that using lesson videos helped them understand the concept of the formal definition of limit. Respondent 1 and Respondent 3 said that the videos helped to systematize their mindsets because they contained steps to solve limit evaluation by formal definition. This point is highlighted by the researcher to be recommended in the lessons in Universitas Serang Raya, especially in Mathematics Education department, which is by making the general flow of proving any material. The development depends on the ability and creativity of each student. At least by making general flow, the students will have an idea on what to do first when facing proving question.

Designing Student Guided Worksheet

Another recommendation to consider to improve learning and student understanding, especially in proving by formal definition, is using the guided worksheet. The purpose of designing the guided worksheet is bridging the concept of formal material and student's mindset to help more systematic thinking (Oktaviyanthi & Dahlan, 2018). The worksheet is a technical realization of the general flow of proving presented theoretically by the lecturer. By designing a guided worksheet, the lecturer can determine how many hints are given and the kind of hint to help the student understand the mindset in solving proving question. Using student worksheet, generally, indicate that there is an increase and positive response from students studying calculus with the help of guided worksheet. Of course, there should be further study to measure the effectiveness of guided worksheet for proving question in the bigger population.

Overview in designing guided worksheet to understand limit evaluation by formal definition can be found from the correct solving strategies by Respondent 1, Respondent 3, and Respondent 4. In general, the steps to prove limit or evaluate the limit by formal definition are presented in Table 1.

Table 1. General Strategy of Evaluating Limit by Formal Definition

Steps	Description	Writing
Stage 1. Preparing the Proof		
1	Determining δ value through the final statement of the formal definition of limit.	$\|f(x) - L\| < \varepsilon$
2	Substituting the given *f(x)* and L values in the question into the point 1 statement.	$\|(\ldots - \cdots) - (\ldots)\| < \varepsilon$
3	Simplifying value in absolute sign in point 2 into $\|x - c\| < \delta$.	$\|\ldots - \cdots\| < \varepsilon$
4	Finishing inequality in point 3 to form $\|x - c\| < \delta$.	$\|\ldots(\ldots - \cdots)\| < \varepsilon$ $\|\ldots\|\|\ldots - \cdots\| < \varepsilon$ $\|\ldots - \cdots\| < \frac{\varepsilon}{\ldots}$
5	Connecting form $\|x - c\| < \delta$ with inequality result from point 4 to find δ value.	$\|\ldots - \cdots\| < \frac{\varepsilon}{\ldots}$ $\|x - c\| < \delta$ to get $\delta = \frac{\varepsilon}{\ldots}$
Stage 2. Proving		
6	For example, $\varepsilon > 0$ is given. This statement is the main point in solving limit proving by formal definition and indicating that our argumentation will apply on every ε value.	$\varepsilon > 0$ is given
7	From the formal definition of limit, it's stated that for every ε value, there's always δ value. Use the determined δ value in point 5.	Define $\delta = \frac{\varepsilon}{\ldots}$
8	The formal definition of limit rigidly states δ value must be consistent, meaning it must be positive. It should be noted that we must select a δ value that meets the definition.	Since $\varepsilon > 0$, then $\delta > 0$
9.1	From the formal definition of limit, for every *x* approaching *c* is denoted as $0 < \|x - c\| < \delta$.	$0 < \|x - c\| < \delta$
9.2	Substitute *c* and δ values in the statement in point 9.1	$\|x - \cdots\| < \frac{\varepsilon}{\ldots}$
10.1	Finish inequality in point j unit it forms the initial inequality in point 2.	$\|\ldots x - \cdots\| < \varepsilon$ $\|(\ldots x - \cdots) - (\ldots)\| < \varepsilon$
10.2	All statements in the formal definition of limit are obtained, so the proving process is finished.	So $\lim_{x \to \_\_} f(x) = \cdots$ is correct.

CONCLUSION

Based on the results and discussion, we could conclude that the respondents used various strategies to solve limit evaluation by formal definition. In general, the systematic of problem solving concluded from the study consisted of two stages, i.e. (1) preparation of proof and (2) proving. The stage of preparing the proof consisted of (1) determining delta value by the final statement of formal definition, (2) substituting the given $f(x)$ and L process, (3) simplifying value in the absolute sign, (4) solving the inequality, and (5) finding the delta value. The stage of proving consisted of (1) stating positive epsilon, (2) defining delta, (3) stating positive delta, (4) substituting the constants and delta values in the initial statement of formal definition, and (5) solving the inequality to create the final inequality statement of the formal definition.

The respondents' common mistakes in solving proving questions by formal definition in this study included (1) poor understanding of implication logical relation which impacted inability to arrange steps of proving, (2) difficulty in processing algebraic form thus inhibiting the proving process, and (3) limited experience in solving proving questions by formal mathematics form.

Some findings which can be recommended to improve learning and understanding are (1) using additional learning media like tutorial video, (2) presenting a schematic proving flow, and (3) guided worksheet. They would be the interesting topics for further research requiring in-depth investigation and study.

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